Multiplication is discontinuous in the Hawaiian earring group (with the quotient topology)

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Abstract. The natural quotient map q from the space of based loops in the Hawaiian earring onto the fundamental group provides a new example of a quotient map such that $q \times q$ fails to be a quotient map. This provides a counterexample to the question of whether the fundamental group (with the quotient topology) of a compact metric space is always a topological group with the standard operations.

1. Introduction

Following the definitions in [**1**], there is a natural quotient topology one can impart on the familiar based fundamental group $\pi_1(X,p)$ of a topological space X.

If $L(X,p)$ denotes the space of p based loops in X with the compact open topology, and if $q : L(X,p) \rightarrow \pi_1(X,p)$ is the natural surjection, endow $\pi_1(X,p)$ with the quotient topology such that $A \subset \pi_1(X,p)$ is closed in $\pi_1(X,p)$ if and only of $q^{-1}(A)$ is closed in $L(X,p)$.

As pointed out in [**6**] and [**3**], it became an open question whether or not group multiplication is always continuous in $\pi_1(X,p)$, various papers have stumbled on the issue, and in the paper at hand we settle the question in the negative via counterexample (Theorem 1) in which X is the compact metric space the Hawaiian earring.

It is well known that outside the category of k-spaces (compactly generated spaces [**10**]) if $q : Y \rightarrow Z$ is a quotient map, the map $q \times q : Y \times Y \rightarrow Z \times Z$ can fail to be a quotient map. However familiar counterexamples of such phenomenon (Ex. 8 p. 141 [**9**]) necessarily involve somewhat strange spaces, since for example it cannot happen that both Y and Z are metrizable.

In this paper we take $X = HE$ to be the Hawaiian earring, the union of a null sequence of circles joined at the common point p.

In the bargain we obtain a naturally occurring example of a quotient map $q : Y \rightarrow Z$ such that $q \times q : Y \times Y \rightarrow Z \times Z$ fails to be a quotient map (Theorem 1).

The example of the Hawaiian earring (a non locally contractible 1 dimensional Peano continuum) is in a sense the simplest counterexample, since in general if X

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is a locally contractible Peano continuum then $\pi_1(X,p)$ is a discrete topological group ([**6**] [**3**]).

Independently Brazas [**2**] has recently discovered some other interesting counterexamples of a different flavor in which X fails to be a regular (T_3) space [**2**].

Moreover the issue of regularity (T_3) also rears now its head as follows. As a consequence of Theorem 1 we know $\pi_1(HE,p)$ is a totally disconnected Hausdorff space but **not** a topological group with the familiar operations. This leads to the open questions "Is $\pi_1(HE,p)$ regular?", "What is the inductive dimension of $\pi_1(HE,p)$?"

2. Main result and implications

The **Hawaiian earring** HE is the union of a null sequence of circles joined at a common point p.

Formally HE is the following subspace of the plane R^2, for an integer $n \geq 1$ let X_n denote the circle of radius $\frac{1}{n}$ centered at $(\frac{1}{n},0)$ and define $HE = \cup_{n=1}^{\infty} X_n$.

Let $p=(0,0)$, let $Y_0=\{p\}$, and let $Y_N = \cup_{n=0}^{N} X_n$.

Let $\lim_{\leftarrow} Y_n$ denote the inverse limit space determined by the natural retractions $r_n : Y_{n+1} \to Y_n$, such that $r_{n|Y_n} = id_{|Y_n}$ and $r_n(X_{n+1}) = \{p\}$.

Thus, (as a subspace with the product topology), $\lim_{\leftarrow} Y_n$ denotes the space of all sequences $(y_0, y_1, y_2, ...) \subset Y_0 \times Y_1 \times Y_2 ...$ such that $y_n = r_n(y_{n+1})$).

Consider the canonical homeomorphism $j : HE \to \lim_{\leftarrow} Y_n$ such that $j(x) = (p,p,...p,x,x,x...)$.

If $\phi : \pi_1(HE,p) \to \lim_{\leftarrow} \pi_1(Y_n,p)$ is the canonical homomorphism induced by j, it is a well known but nontrivial fact that ϕ is one to one [**8**].

Moreover as shown in ([**4**] [**5**]), elements of $\pi_1(HE,p)$, can be understood precisely as the 'irreducible' infinite words in the symbols $\{x_1, x_2, ..\}$ such that no symbol appears more than finitely many times.

In particular $\pi_1(HE,p)$ naturally contains a subgroup canonically isomorphic to the free group over symbols $\{x_1, x_2, ..\}$.

Let $L(HE,p)$ denote the space of maps $f : [0,1] \to HE$ such that $f(0) = f(1) = p$ and impart $L(HE,p)$ with the topology of uniform convergence.

Let $q : L(HE,p) \to \pi_1(HE,p)$ denote the canonical quotient map such that $q(f) = q(g)$ if and only if f and g are path homotopic in HE.

Let $q_n = (\frac{2}{n},0) \in X_n$. Define the **oscillation number** $O_n : L(HE,p) \to \{0,1,2,3,4,...\}$ to be the maximum number m such that there exists a set $T = \{0, t_1, t_2, ..., t_{2m}\} \subset [0,1]$ such that $0 < t_1 < t_2 .. < t_{2m} = 1$ with $f(t_{2i}) = p$ and $f(t_{2i+1}) = q_n$.

Uniform continuity of f over $[0,1]$ ensures that $O_n(f) < \infty$. Other elementary properties of O_n are observed or established in the following remarks. (See also [**7**]).

Remark 1. *Suppose $f_k \to f$ uniformly in $L(HE,p)$ and $O_n(f_k) \geq m$ as shown by the sets $T_k \subset [0,1]$ such that $|T_k| = 2m+1$. Then if $T \subset [0,1]$ is a subsequential limit of $\{T_k\}$ in the Hausdorff metric, then T shows $O_n(f) \geq m$.*

Remark 2. *Suppose f and g are in the same path component of $L(HE,p)$ and suppose $g : [0,1] \to Y_m$ is a geodesic corresponding to a maximally reduced finite word w in the free group F_m on m letters. Then $O_n(f) \geq O_n(g)$. (Replacing f by $f_1 = r_m(f)$ we have $O_n(f) \geq O_n(f_1)$. For each nontrivial interval $J \subset im(f_1) \backslash \{p\}$*

such that $f_{1\overline{J}}$ is an inessential based loop, replace f_J by the constant map to obtain f_2 and observe $O_n(f_2) \leq O_n(f_1)$ and let $v \in F_m$ denote the (unreduced) finite word corresponding to f_2. Then $|v| \geq |w|$ and hence $O_n(f_2) \geq O_m(g)$.)

Remark 3. *Since $\phi : \pi_1(HE,p) \to \lim_{\leftarrow} \pi_1(Y_n,p)$ is one to one, the path components of $L(HE,p)$ are closed subspaces of $L(HE,p)$. (If $\{f_n\}$ is a sequence of homotopic based loops in HE, and if $f_n \to f$ and then $r_N(f_n) \to r_N(f)$ for each N. Since Y_N is locally contractible $r_N(f_n)$ is eventually path homotopic to $r_N(f)$. Thus, since ϕ is one to one, f is path homotopic to f_1 in HE). See also the proof of Theorem 2* [**7**].

Remark 4. *If Z is a metric space such that each path component of Z is a closed subspace of Z, then each path component of $Z \times Z$ is a closed subspaces of $Z \times Z$. (If $(x_n, y_n) \to (x,y)$ and $\{(x_n,y_n)\}$ is in a path component of $Z \times Z$ then obtain paths α and β in Z connecting x to $\{x_n\}$ and y to $\{y_n\}$ and (α,β) is the desired path in $Z \times Z$).*

What follows is the main result of the paper.

Theorem 1. *The product of quotient maps $q \times q : L(HE,p) \times L(HE,p) \to \pi_1(HE,p) \times \pi_1(HE,p)$ fails to be a quotient map, standard multiplication (by path class concatenation) $M : \pi_1(HE,p) \times \pi_1(HE,p) \to \pi_1(HE,p)$ is discontinuous, and the fundamental group $\pi_1(HE,p)$ fails to be a topological group with the standard group operations.*

Proof. Let $x_n \in L(HE,p)$ orbit X_n once counterclockwise.

Applying path concatenation, for integers $n \geq 2$ and $k \geq 2$ let $a(n,k) \in L(HE,p)$ be a based loop corresponding to the finite word $(x_n x_k x_n^{-1} x_k^{-1})^{k+n}$ and let $w(n,k) \in L(HE,p)$ be a based loop corresponding to the finite word $(x_1 x_k x_1^{-1} x_k^{-1})^n$.

Let $F \subset \pi_1(HE,p) \times \pi_1(HE,p)$ denote the set of all doubly indexed ordered pairs $([a(n,k)],[w(n,k)])$.

Let $P \in L(HE,p)$ denote the constant map such that $f([0,1]) = \{p\}$.

To prove $q \times q$ fails to be a quotient map it suffices to prove that F is not closed in $\pi_1(HE,p) \times \pi_1(HE,p)$ but that $(q \times q)^{-1}(F)$ is closed in $L(HE,p) \times L(HE,p)$.

To prove F is not closed in $\pi_1(HE,p) \times \pi_1(HE,p)$ we will prove that $([P],[P]) \notin F$ but $([P],[P])$ is a limit point of F.

Recall $\phi : \pi_1(HE,p) \to \lim_{\leftarrow} \pi_1(Y_n,p)$ is one to one and $k \geq 2$. Thus $[P] \neq [w(n,k)]$ and $[P] \neq [a(n,k)]$. Thus $([P],[P]) \notin F$.

Suppose $[P] \in U$ and U is open in $\pi_1(HE,p)$. Let $V = q^{-1}(U)$. Then V is open in $L(HE,p)$ since, by definition, q is continuous.

Note $P \in V$. Thus there exists N and K such that if $n \geq N$ and $k \geq K$ then $a(n,k) \in V$.

Note $(x_1 x_1^{-1})^N$ is path homotopic to P and hence $(x_1 x_1^{-1})^N \in V$. Note (suitably parameterized over $[0,1]$), $w(N,k) \to (x_1 x_1^{-1})^N$ uniformly in $L(HE,p)$.

Thus there exists $K_2 \geq K$ such that if $k \geq K_2$ then $w(N,k) \in V$. Hence $([w(N,K_2)],[a(N,K_2)]) \in U \times U$.

This proves $([P],[P])$ is a limit point of F, and thus F is not closed in $\pi_1(HE,p) \times \pi_1(HE,p)$.

To prove $(q \times q)^{-1}(F)$ is closed in $L(HE,p) \times L(HE,p)$ suppose $(f_m, g_m) \to (f,g)$ uniformly and $(f_m,g_m) \in (q \times q)^{-1}(F)$.

Note $O_1(w(n,k)) = 2n$ and $O_N(a(N,k)) \geq 2(N+k)$.

Let $a(n_m,k_m)$ and $w(n_m,k_m)$ be path homotopic to respectively f_m and g_m.

By Remark 2 $O_1(g_m)\geq O_1(w(n_m,k_m))=2n_m$.

Thus if $\{n_m\}$ contains an unbounded subsequence then, by Remark 1, $O_1(g)\geq \limsup O_1(w(n_m,k_{n_m}))=\infty$ and we have a contradiction since $O_1(g)<\infty$.

Thus $\{n_m\}$ is bounded and the sequence $\{n_m\}$ takes on finitely many values.

In similar fashion, if $\{k_m\}$ is unbounded then there exists N and a subsequence $\{k_{m_l}\}$ such that $O_N(a(N,k_{m_l})\rightarrow\infty$. Thus $O_N(f)\geq \limsup O_N(a(N,k_{m_l}))=\infty$ contradicting the fact that $O_N(f)<\infty$.

Thus both $\{n_m\}$ and $\{k_m\}$ are bounded and hence (by the pigeon hole principle) there exists a path component $A\subset L(HE,p)\times L(HE,p)$ containing a subsequence (f_{m_l},g_{m_l}).

It follows from Remarks 3 and 4 that $(f,g)\in A$.

Thus $(q\times q)^{-1}(F)$ is closed and hence $q\times q$ fails to be a quotient map.

In similar fashion we will prove that group multiplication $M:\pi_1(HE,p)\times \pi_1(HE,p)\rightarrow \pi_1(HE,p)$ is discontinuous, and hence $\pi_1(HE,p)$ will fail to be a topological group with the standard group operations. To achieve this we will exhibit a closed set $A\subset \pi_1(HE,p)$ such that $M^{-1}(A)$ is not closed in $\pi_1(HE,p)\times \pi_1(HE,p)$.

Consider the doubly indexed set $A=M(F)\subset \pi_1(HE,p)$ such that each element of A is of the form $[a(n,k)]*[w(n,k)]$ (with $*$ the denoting familiar path class concatenation).

On the one hand observe by definition (and since ϕ is one to one) $[a(n,k)]*[w(n,k)]\neq [P]$.

Thus $[P]\notin A$ and $([P],[P])\notin M^{-1}(A)$. Note $F\subset M^{-1}(A)$ and by the previous argument $([P],[P])$ is a limit point of F. Thus $M^{-1}(A)$ is not closed in $\pi_1(HE,p)\times \pi_1(HE,p)$.

On the other hand we will prove A is closed in $\pi_1(HE,p)$ by proving $q^{-1}(A)$ is closed in $L(HE,p)$. Suppose $f_m\rightarrow f\in L(HE,p)$ and $f_m\in q^{-1}(A)$.

Obtain n_m and k_m such that $f_m\in [a(n,k)]*[w(n,k)]$.

In similar fashion to the previous proof, if $\{n_m\}$ is unbounded we obtain the contradiction $O_1(f)\geq \limsup O_1(f_m)=\infty$.

If $\{n_m\}$ is bounded and $\{k_m\}$ is unbounded we obtain N and a subsequence k_{m_l} and the contradiction $O_N(f)\geq \limsup O_N(f_{m_l})=\infty$.

Thus both $\{n_m\}$ and $\{k_m\}$ are bounded. It follows by the pigeon hole principle that some path component $B\subset L(HE,p)$ contains a subsequence $\{f_{m_l}\}$ and it follows from Remark 3 that $f\in B$. Hence $q^{-1}(A)$ is closed in $L(HE,p)$ and thus A is closed in $\pi_1(HE,p)$. $\square$

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